Approximately holomorphic techniques in symplectic topology

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Symplectic manifolds

A symplectic structure on a smooth manifold is a 2-form ω such that $d\omega = 0$ and $\omega \wedge \cdots \wedge \omega$ is a volume form.

Example: $\mathbb{R}^{2n}$, $\omega_0 = \sum dx_i \wedge dy_i$.

(Darboux: every symplectic manifold is locally $\simeq (\mathbb{R}^{2n}, \omega_0)$, i.e. there are no local invariants).

Example: Riemann surfaces (Σ, vol_Σ) are symplectic.

Example: Every Kähler manifold is symplectic.
(includes all complex projective manifolds)

but the symplectic category is much larger.
(Gompf 1994: $\forall G$ finitely presented group, $\exists (X^4, \omega)$ compact symplectic such that $\pi_1(X) = G$).

Symplectic manifolds are not always complex, but they are almost-complex, i.e. there exists $J \in \mathrm{End}(TX)$ such that

$$J^2 = -\mathrm{Id}, \quad g(u,v) := \omega(u, Jv) \text{ Riemannian metric.}$$

At any given point (X, ω, J) looks like $(\mathbb{C}^n, \omega_0, i)$, but J is not integrable ($\nabla J \neq 0$; $\bar\partial^2 \neq 0$; $[T^{1,0}, T^{1,0}] \not\subset T^{1,0}$). So there are no holomorphic functions (in particular no holomorphic local coordinates).

The moduli space of compatible almost-complex structures is always contractible.

Symplectic topology

Typical problems:

– Which smooth manifolds admit symplectic structures ?

– Classify symplectic structures on a given smooth manifold.

(Moser: if $[\omega] \in H^2(X, \mathbb{R})$ is fixed then all small deformations are trivial).

Why we care:

– Physics (classical mechanics; string theory; ...)

– Next step after understanding complex manifolds.

Some facts from complex geometry extend to symplectic manifolds; most don't.

A lot is known if $\dim X = 4$. Core ingredient: structure of Seiberg-Witten / Gromov-Witten invariants of symplectic 4-manifolds (Taubes).

For $\dim X \geq 6$, almost nothing is known. E.g., no known non-trivial obstruction to the symplecticity of compact 6-manifolds (except $\exists [\omega] \in H^2(X, \mathbb{R})$ s.t. $[\omega]^{\wedge 3} \neq 0$).

Approximately holomorphic geometry

Idea:

Since we have almost-complex structures, even though there are no holomorphic sections and linear systems, we can work similarly with approximately holomorphic objects.

(Donaldson, ∼1995)

Setup: (X^{2n}, ω) symplectic, compact

- $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$ (not restrictive)
- J compatible with ω ; $g(.,.) = \omega(., J.)$
- L line bundle such that $c_1(L) = \frac{1}{2\pi}[\omega]$
- ∇^L, with curvature $-i\omega$; $\nabla^L = \partial^L + \bar{\partial}^L$.

$$\bar{\partial}^L s(v) = \tfrac{1}{2}(\nabla^L s(v) + i\nabla^L s(Jv)).$$

If X Kähler, then L is a holomorphic ample line bundle, i.e. $L^{\otimes k}$ has many holomorphic sections for k large enough.

$\Rightarrow$ projective embeddings $X \hookrightarrow \mathbb{CP}^N$ (Kodaira).

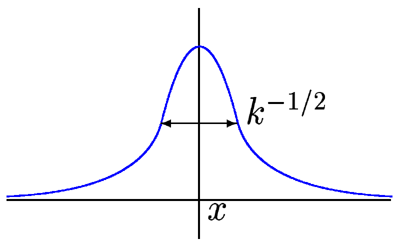
$\Rightarrow$ smooth hypersurfaces (Bertini).

$\Rightarrow$ linear systems, projective maps.

Approximately holomorphic sections

X symplectic: J is not integrable $\Rightarrow$ no holomorphic sections.
However, local approximately holomorphic model:

$$(X,x),\ \omega,\ J \longleftrightarrow (\mathbb{C}^n,0),\ \omega_0,\ (i+\dots)$$
$$L^{\otimes k},\ \nabla \longleftrightarrow \underline{\mathbb{C}},\ d+\tfrac{k}{4}\sum(z_j d\bar{z}_j - \bar{z}_j dz_j).$$



$\Rightarrow$ $s_{k,x}(z) = \exp(-\frac{1}{4}k|z|^2)$ is approx. holomorphic !

A sequence of sections $s_k \in \Gamma(L^{\otimes k})$ is approx. holomorphic if $\sup|\bar\partial s_k| < C\,k^{-1/2} \sup|\partial s_k|$ (+ similarly for higher order derivatives).

(open condition! $\Rightarrow$ no finite dim. space of sections)

For $k \gg 0$ the curvature of $L^{\otimes k}$ ($F_k = -ik\omega$) probes the small-scale geometry of $X \Rightarrow J$ becomes almost integrable.
($\sup|\partial s_k| \sim \sqrt{k}$: rescale metric by $\sqrt{k}$ for uniform bounds)

Goal: find some approx. holomorphic sections which behave "generically".

Approximately holomorphic hypersurfaces

Theorem 1. (Donaldson, 1996) *If $k \gg 0$, then $L^{\otimes k}$ admits approx. holomorphic sections s_k whose zero sets W_k are smooth symplectic hypersurfaces.*

Make up for loss of holomorphicity by achieving estimated transversality: require $|\partial s_k(x)| \gg \sup |\bar{\partial} s_k|$ along $s_k^{-1}(0)$. (uniform lower bound instead of just $\partial s_k(x) \neq 0$)

These symplectic submanifolds have some special properties typical of complex submanifolds:

– Lefschetz hyperplane: W_k have the same homotopy and homology groups as X up to middle dimension.

– Uniqueness: fixing $k \gg 0$, the submanifolds W_k are, up to isotopy, independent of all choices made (even for J !).

Also consider linear systems of ≥ 2 sections:

E.g., (s_0, s_1) well-chosen approx. hol. sections of $L^{\otimes k}$ $(k \gg 0)$
$\Rightarrow$ symplectic Lefschetz pencils (Donaldson, 1999)
(= family of hypersurfaces parametrized by $\mathbb{CP}^1$ with isolated singularities and standard local models).

The topological data encoded in the pencil determines (X, ω) up to symplectomorphism.

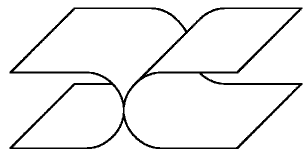
Branched covers of $\mathbb{CP}^2$

Theorem 2. (A., 2000) *For $k \gg 0$, three suitable approx. hol. sections of $L^{\otimes k}$ define a map $X \to \mathbb{CP}^2$ with generic local models, canonical up to isotopy.*

(X^4, ω) symplectic, $s_0, s_1, s_2 \in \Gamma(L^{\otimes k})$ well-chosen
$\Rightarrow f = (s_0 : s_1 : s_2) : X \to \mathbb{CP}^2$.

Local models near branch curve $R \subset X$:

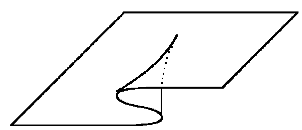
– branched cover : $(x, y) \mapsto (x^2, y)$.



$R: \ x = 0 \qquad f(R): \ X = 0$

$X^{2n} \to \mathbb{CP}^2$: $(z_1, \ldots, z_n) \mapsto (z_1^2 + \cdots + z_{n-1}^2, z_n)$

– cusp : $(x, y) \mapsto (x^3 - xy, y)$.



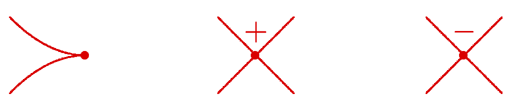
$R: \ y = 3x^2 \qquad f(R): \ 27X^2 = 4Y^3$

$X^{2n} \to \mathbb{CP}^2$: $(z_1, \ldots, z_n) \mapsto (z_1^3 - z_1 z_n + z_2^2 + \cdots + z_{n-1}^2, z_n)$

R smooth connected symplectic curve in X.
$D = f(R)$ symplectic, immersed except at the cusps.

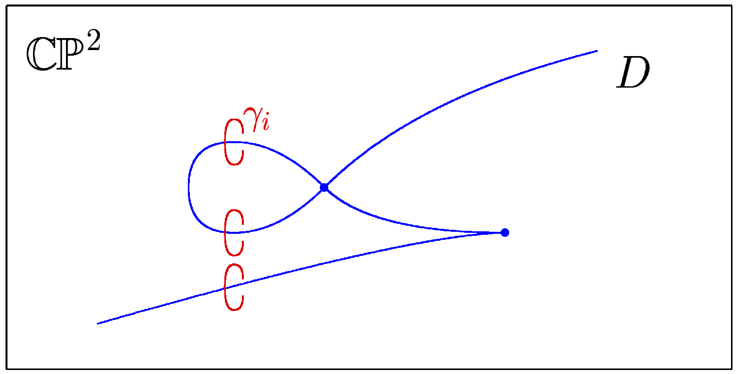
Generic singularities :

complex cusps; nodes (both orientations)



Theorem 2 $\Rightarrow$ up to cancellation of nodes, the topology of D is a symplectic invariant (if k large).

Topological invariants

$X \xrightarrow{N:1}$ 

Topological data for a branched cover of $\mathbb{CP}^2$:

1) Branch curve: $D \subset \mathbb{CP}^2$
(up to isotopy and node cancellations).

2) Monodromy: $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N \quad (N = \deg f)$
(surjective, maps γ_i to transpositions).

D and θ determine (X, ω) up to symplectomorphism.

When $\dim X > 4$, main difference: θ takes values in the mapping class group of the generic fiber.

This group is complicated; however there is a dimensional induction procedure $\Rightarrow$ given (X^{2n}, ω) and $k \gg 0$ we get

1) $(n-1)$ plane curves $D_n, D_{n-1}, \dots, D_2 \subset \mathbb{CP}^2$.
2) $\theta_2 : \pi_1(\mathbb{CP}^2 - D_2) \to S_N$.

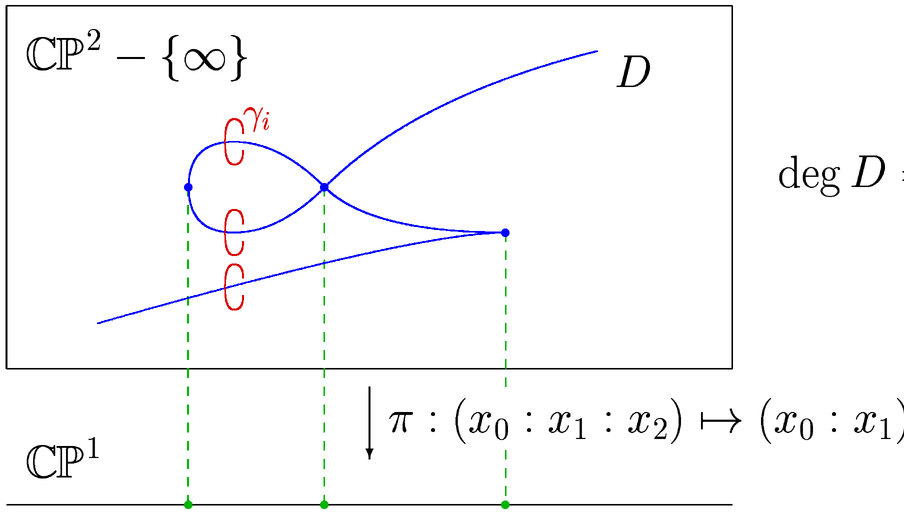
and these data determine (X, ω) up to symplectomorphism.

$\Rightarrow$ In principle it is enough to understand plane curves !

The topology of plane curves

(Moishezon-Teicher, Auroux-Katzarkov-Yotov)

Perturbation $\Rightarrow D =$ singular branched cover of $\mathbb{CP}^1$.



$\deg D = d$

Monodromy $= \rho : \pi_1(\mathbb{C} - \{\text{pts}\}) \to B_d$ (braid group)

$\Rightarrow D$ is described by a "braid group factorization" (involving cusps, nodes, tangencies).

The braid factorization characterizes D completely.

Problem: once computed, cannot be compared.

$\Rightarrow$ more manageable (incomplete) invariant ?

Fundamental groups of complements

(Moishezon-Teicher, Auroux-Donaldson-Katzarkov)

Test problem: distinguish symplectically some homeomorphic complex surfaces of general type. (Seiberg-Witten etc. are useless for this).

Moishezon-Teicher: use $\pi_1(\mathbb{CP}^2 - D)$ as invariant.

$\pi_1(\mathbb{CP}^2 - D)$ is generated by "geometric generators" $(\gamma_i)_{1 \le i \le d}$; relations given by the braid factorization.

Problem: in the symplectic case, node cancellations affect $\pi_1(\mathbb{CP}^2 - D_k)$. $\Rightarrow$ consider a quotient $G_k = \pi_1(\mathbb{CP}^2 - D_k)/\sim$ that is a symplectic invariant for $k \gg 0$.

Fact: $1 \to G_k^0 \to G_k \to S_N \times \mathbb{Z}_d \to \mathbb{Z}_2 \to 1.$
$(N = \deg f_k,\ d = \deg D_k)$

Known examples ($\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, ruled surfaces, double covers, ...): for large k,

1) $G_k = \pi_1(\mathbb{CP}^2 - D_k)$.
2) G_k^0 is almost abelian: $[G_k^0, G_k^0]$ has at most 4 elements.
3) ... but $\mathrm{Ab}(G_k^0)$ depends only on homeomorphism data! (intersection pairing, divisibility of $[\omega]$ and K_X)