Branched coverings of $\mathbb{CP}^2$ and invariants of symplectic 4-manifolds

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1) Geometry
2) Topology (joint with L. Katzarkov)

Introduction

X compact Kähler manifold, L ample bundle.

Holomorphic sections of L^k, $k \gg 0$

$\Rightarrow$ projective embedding $X \hookrightarrow \mathbb{CP}^N$ (Kodaira).

$\Rightarrow$ smooth hypersurfaces (Bertini).

$\Rightarrow$ ...

X complex surface, 3 generic sections of L^k
$\Rightarrow f : X \to \mathbb{CP}^2$ branched covering,

$$\text{singularities} = \text{cusps} + \text{nodes}.$$

(X^{2n}, ω) compact symplectic manifold :
$\exists J$ compatible almost-complex structure.

J is not integrable
$\Rightarrow$ no holomorphic coordinates
$\Rightarrow$ no holomorphic sections

Donaldson's idea :

Approximately holomorphic sections
$\Rightarrow$ symplectic analogues of classical results.

Asymptotically holomorphic sections

(X^{2n}, ω) symplectic, compact

- $\frac{1}{2\pi}[\omega] \in H^2(X, \mathbb{Z})$ (not restrictive)
- J compatible with ω ; $g(.,.) = \omega(., J.)$
- L line bundle such that $c_1(L) = \frac{1}{2\pi}[\omega]$
- $|\cdot|_L$; ∇^L, curvature $-i\omega$
- $g_k = k\,g$.

Definition. $(s_k)_{k\gg 0} \in \Gamma(E_k)$ *are asymptotically holomorphic ("A.H.") if*

$$\forall p \in \mathbb{N},\ |s_k|_{C^p, g_k} = O(1) \quad \textit{and} \quad |\bar\partial s_k|_{C^p, g_k} = O(k^{-1/2}).$$

Definition. $(s_k)_{k\gg 0} \in \Gamma(E_k)$ *are uniformly transverse to* 0 *if* $\exists \eta > 0 \,/\, s_k$ *is* η*-transverse to* 0 $\forall k$, *i.e.*

$$\forall x \in X,\ |s_k(x)| < \eta \Rightarrow \nabla s_k(x) \textit{ surjective and } > \eta.$$

Proposition. *Let* $(s_k)_{k\gg 0} \in \Gamma(E_k)$, A.H. *and uniformly transverse to* 0 *: then for* $k \gg 0$, $W_k = s_k^{-1}(0)$ *is a symplectic submanifold of* X *(approximately* J*-holomorphic).*

Symplectic submanifolds and beyond

Theorem 1 (Donaldson) *For $k \gg 0$, the bundles L^k admit sections which are* A.H. *and uniformly transverse to* 0.

$\Rightarrow$ construction of symplectic submanifolds.

Theorem 2 (Donaldson) *For $k \gg 0$, the bundles L^k admit pairs of* A.H. *sections which endow X with a structure of symplectic Lefschetz pencil.*

Structure of the proof

1. existence of very localized A.H. sections of L^k
2. effective Sard theorem for A.H. functions :
   $\Rightarrow$ get uniform transversality over a small ball.
3. globalization principle
   (transversality is an open property).

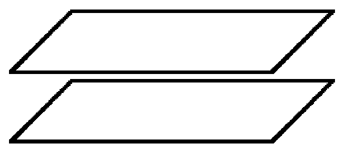
Branched coverings

$$\dim X = 4: \text{ nowhere vanishing section of } \mathbb{C}^3 \otimes L^k$$
$$\Rightarrow \quad f = (s^0 : s^1 : s^2) : X \to \mathbb{CP}^2.$$

Definition. *A map $f : X \to \mathbb{CP}^2$ is ϵ-holomorphically modelled at x on $g : \mathbb{C}^2 \to \mathbb{C}^2$ if $\exists U \ni x$, $V \ni f(x)$, and local C^1-diffeomorphisms $\phi : U \to \mathbb{C}^2$ and $\psi : V \to \mathbb{C}^2$, ϵ-holomorphic, (i.e. $|\phi_* J - \mathbb{J}_0| < \epsilon$) such that $f_{|U} = \psi^{-1} \circ g \circ \phi$.*

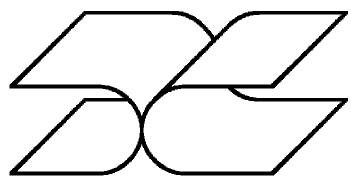
Definition. *A map $f : X \to \mathbb{CP}^2$ is an ϵ-holomorphic covering branched along $R \subset X$ if Df is surjective everywhere except along R, and if f is locally ϵ-holomorphically modelled at any point of X on one of the following maps :*

– *local diffeomorphism : $(x, y) \mapsto (x, y)$.*



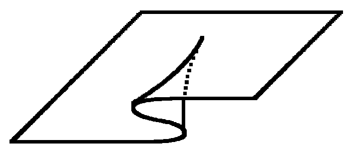
– *branched covering : $(x, y) \mapsto (x^2, y)$.*

$R: \ x = 0 \qquad f(R): \ X = 0$



– *cusp : $(x, y) \mapsto (x^3 - xy, y)$.*

$R: \ y = 3x^2 \qquad f(R): \ 27X^2 = 4Y^3$



Existence of branched coverings

Theorem 3. *For $k \gg 0$, there exist* A.H. *sections of $\mathbb{C}^3 \otimes L^k$ which make X an ϵ_k-holomorphic branched covering of $\mathbb{CP}^2$, with $\epsilon_k = O(k^{-1/2})$.*

Topological properties $\rightsquigarrow$ analytic properties ?

Transversality conditions :

$s_k \in \Gamma(\mathbb{C}^3 \otimes L^k)$ A.H., $f_k = \mathbb{P}(s_k)$, $\gamma > 0$ fixed.

(T1) $|s_k(x)| \geq \gamma \ \forall x \in X$.

(T2) $|\partial f_k(x)|_{g_k} \geq \gamma \ \forall x \in X$.

Branching $\equiv$ $(2,0)$-Jacobian $\mathrm{Jac}(f_k) = \det(\partial f_k)$.

(T3) $\mathrm{Jac}(f_k)$ is γ-transverse to 0.

$\Rightarrow R(s_k) = \mathrm{Jac}(f_k)^{-1}(0)$ symplectic and smooth.

Angle between $TR(s_k)$ and $\mathrm{Ker}\,\partial f_k \ \rightsquigarrow \ \mathcal{T}(s_k)$.

(T4) $\mathcal{T}(s_k)$ is γ-transverse to 0.

$\Rightarrow$ zeros of $\mathcal{T}(s_k)$ = isolated, non-degenerate cusps

Holomorphic case : (T1–T4) $\Rightarrow$ branched covering.
Vanishing of $\bar{\partial} f_k$ at the branch points ?

J-compatibility conditions :

$\exists\, \tilde{J}_k$ compatible with ω, integrable near the cusps and satisfying $|\tilde{J}_k - J| = O(k^{-1/2})$, such that

(C1) f_k is $\tilde{J}_k$-holomorphic near the cusps.

(C2) $\forall x \in R_{\tilde{J}_k}(s_k)$, Ker $\partial f_k(x) \subset$ Ker $\bar{\partial} f_k(x)$.

Proposition. $(s_k)_{k\gg 0} \in \Gamma(\mathbb{C}^3 \otimes L^k)$, A.H. , *satisfying (T1–T4) and (C1–C2)* $\Rightarrow$ *for* $k \gg 0$, $f_k = \mathbb{P}(s_k)$ *is an* ϵ_k*-holomorphic branched covering,* $\epsilon_k = O(k^{-1/2})$.

$\Rightarrow$ existence of sections satisfying (T1–T4) & (C1–C2) ?

– (T1–T4) : techniques $\simeq$ construction of submanifolds.

- local transversality result : very localized perturbation of $s_k \rightsquigarrow$ property over a small ball.
- globalization principle : combine the local perturbations $\rightsquigarrow$ property at any point of X.

– (C1–C2) : small perturbations near $R(s_k)$
$\Rightarrow$ add to s_k a quantity which exactly cancels $\bar{\partial} f_k$.

Characterization of symplectic manifolds

Properties of constructed coverings w.r.t. the symplectic structure ?

Proposition. *The 2-forms* $\tilde{\omega}_t = t\, f^*\omega_0 + (1-t)\, k\omega$ *are symplectic* $\forall t \in [0,1[$*, and* $(X, \tilde{\omega}_t)$ *is then symplectomorphic to* $(X, k\omega)$*.*

The property of being a branched covering of $\mathbb{CP}^2$ characterizes symplectic manifolds in dimension 4 :

Proposition. *Let* $f : M^4 \to \mathbb{CP}^2$ *be a map which identifies at any point with one of the three models for branched coverings in local coordinates (*A.H. *chart on* $\mathbb{CP}^2$*, but not on* M*).*

Then M *admits a symplectic structure arbitrarily close to* $f^*\omega_0$ *in its cohomology class. This symplectic structure is canonical up to symplectomorphism.*

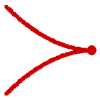
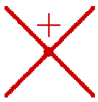
Coverings and symplectic invariants

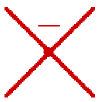
Theorem 4. *For $k \gg 0$, the branched coverings obtained from* A.H. *sections of $\mathbb{C}^3 \otimes L^k$ are unique up to isotopy, independently of the chosen J.*

$\Rightarrow$ symplectic invariants of (X, ω).

$D = f(R) \subset \mathbb{CP}^2$ is a symplectic curve.

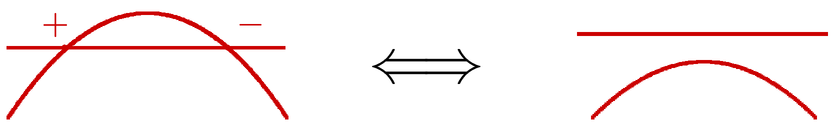
Generic singularities :

1.  cusps.
2. 

nodes with positive transverse intersection.
3. 

nodes with negative transverse intersection.

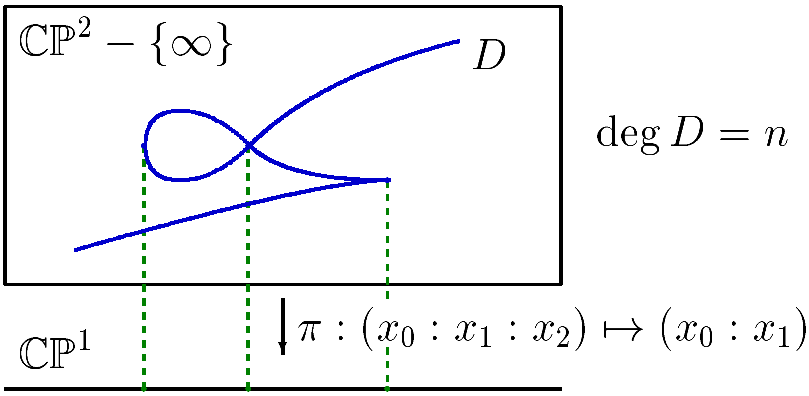
Theorem 4 $\Rightarrow$ up to cancellation of nodes, the topology of D is a symplectic invariant.



$\Rightarrow$ extension of Moishezon and Teicher's braid group techniques to the symplectic case.

Monodromy and braid groups

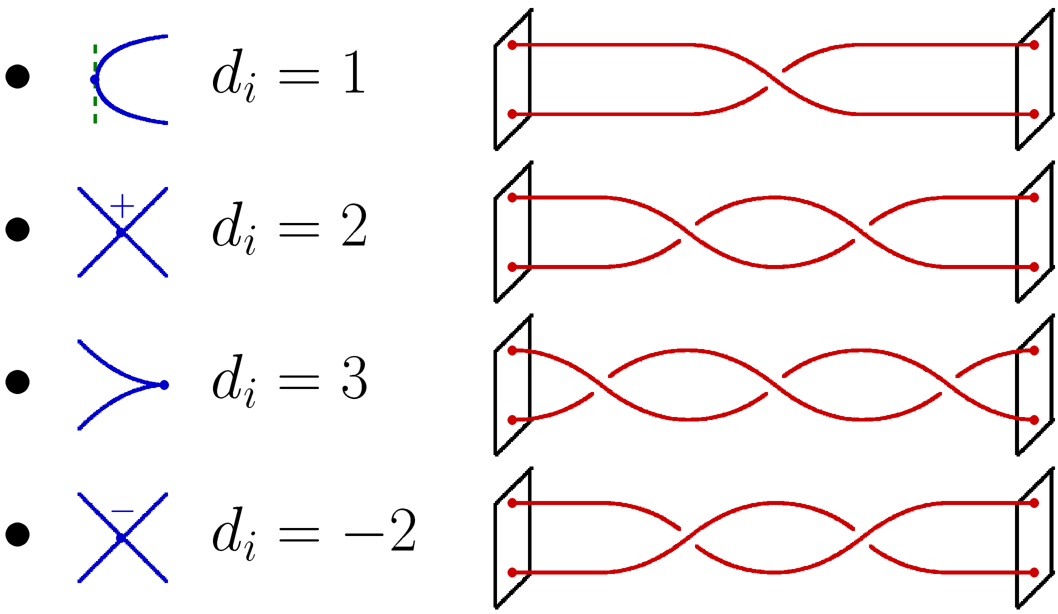
After perturbation, the curve D can be realized as a singular branched covering of $\mathbb{CP}^1$.



Fiber $\simeq \mathbb{C} \;\Rightarrow$ restricting to $\mathbb{C}^2 = \pi^{-1}(\mathbb{C})$, monodromy with values in the braid group B_n :

$$\rho : \pi_1(\mathbb{C} - \mathrm{crit}) \to B_n.$$

The topology of D is described by a braid group factorization, $\Delta^2 = \prod Q_i\, X_1^{d_i}\, Q_i^{-1}$, $d_i \in \{-2, 1, 2, 3\}$:



Up to conjugation, Hurwitz moves and node eliminations, this factorization is a symplectic invariant.

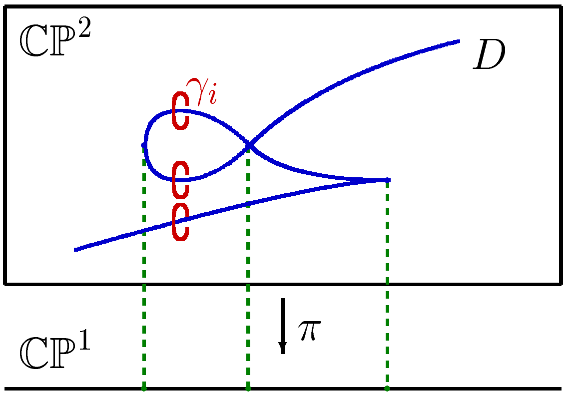
Reconstructing a symplectic 4-manifold

Algebraic data characterizing a branched covering :

1. Braid factorization $\Delta^2 = \prod Q_i \, X_1^{d_i} \, Q_i^{-1}$.
2. Geometric monodromy representation

$$\theta : \pi_1(\mathbb{CP}^2 - D) \twoheadrightarrow S_N.$$

$\pi_1(\mathbb{CP}^2 - D)$ is generated by "geometric generators" $(\gamma_i)_{1\le i\le n}$; relations given by the braid factorization.



θ maps geometric generators to transpositions.
cusp $\Rightarrow$ $(12)(23)$, node $\Rightarrow$ $(12)(34)$.

Theorem 5. *The braid factorization Δ^2 determines D up to smooth isotopy ; D and θ determine (X, ω) up to symplectic isotopy.*

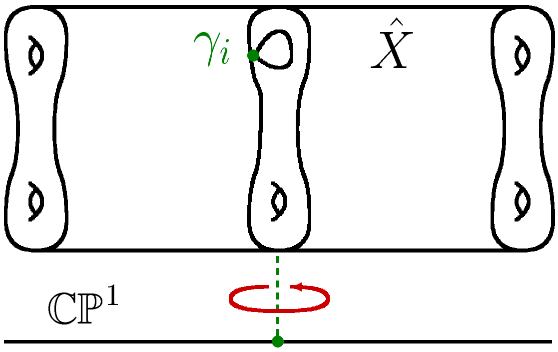
Branched coverings and Lefschetz pencils

(X^4, ω)

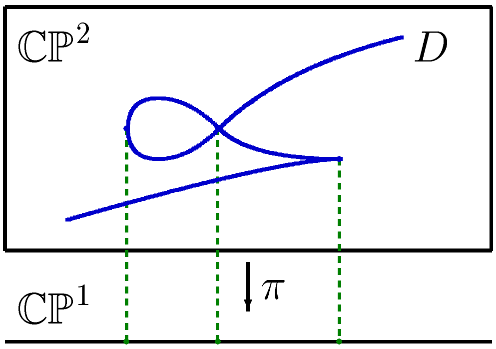
(Donaldson)

Symplectic Lefschetz pencil

Branched covering



monodromy = Dehn twist



Factorization in the mapping class group

$Id = \prod_i t_{\gamma_i}$

Factorization in the braid group

$\Delta^2 = \prod_i Q_i X_1^{d_i} Q_i^{-1}$

+ monodromy repn. θ.

?

Branched coverings and Lefschetz pencils

1. By forgetting one of the components (i.e. projecting to $\mathbb{CP}^1$), a branched covering becomes a symplectic Lefschetz pencil.

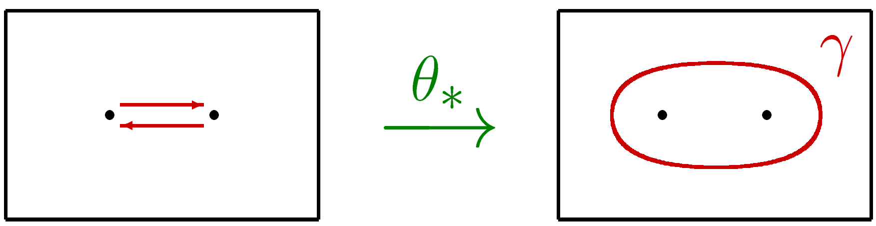
   $\Rightarrow$ alternate proof of Donaldson's result.

2. $\theta : \pi_1(\mathbb{CP}^2 - D) \to S_N$ determines a subgroup $B_n^0(\theta) \subset B_n$ and a group homomorphism

$$\theta_* : B_n^0(\theta) \to \mathrm{Map}_g.$$

$B_n^0(\theta)$ contains the image of the braid monodromy.

- the factors of degree ± 2 or 3 lie in the kernel of θ_*.
- θ_* maps the factors of degree 1 to Dehn twists.



half-twist

Dehn twist along a lift of γ

$\Rightarrow$ Δ^2 and θ allow the explicit computation of the monodromy of the corresponding Lefschetz pencil.